ECON 6376 Time Series Analysis Q2PS2

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1 Solution of Question 2

Derivation of the AR2 process moments and ACF.

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

where ε_t is a white noise and $\varepsilon_t \sim N(0, \sigma^2)$

It means $\forall$ $E(\varepsilon_t \varepsilon_s) = \sigma^2$ if $t = s$; $E(\varepsilon_t \varepsilon_s) = 0$ otherwise.

1.1 First Moment

For the first moment, we could just take a Expectation for the both sides of the equation, since the random variables are linearly combined.

$$\begin{aligned} E(X_t) &= E(\phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t) \\ &= \phi_0 + \phi_1 E(X_{t-1}) + \phi_2 E(X_{t-2}) + E(\varepsilon_t) \end{aligned} \tag{1}$$

Since $E(\varepsilon_t) = 0$ by given, and we assume AR2 is a stationary process for now, so $E(X_t) = E(X_{t-1}) = E(X_{t-2}) = \mu$

Substitute those conditions in the equation (1) above, we have

$$\mu = \phi_0 + \phi_1 \mu + \phi_2 \mu$$

Rearrange and solve out the equation,

$$\mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

To reconcile with the assumption of the stationary process, $\phi_1 + \phi_2 < 1$

1.2 Second Moment

For the second moment, we could just take a Variation for the both sides of the equation, since the random variables are linearly combined.

$$Var(X_t) = Var(\phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t) \tag{2}$$

But we need to be cautious of the covariances of the terms. By using the covariance formula to decompose the right hand side of the equation 2.

$$Cov(aX_t, bY_t) = a^2Var(X_t) + b^2Var(Y_t) + 2abCov(X_t, Y_t)$$

So we have,

$$Var(X_t) = Var(\phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t)$$
$$= Var(\phi_0) + Var(\phi_1 X_{t-1}) + Var(\phi_2 X_{t-2}) + Var(\varepsilon_t) + 2\phi_1\phi_2 Cov(X_{t-1}, X_{t-2}) \qquad (3)$$

Since

$$Cov(\phi_0, X_{t-1}) = Cov(\phi_0, X_{t-2}) = Cov(\phi_0, \varepsilon_t) = Cov(X_{t-1}, \varepsilon_t) = Cov(X_{t-2}, \varepsilon_t) = 0$$

Continue equation (3),

$$Var(X_t) = Var(\phi_0) + Var(\phi_1 X_{t-1}) + Var(\phi_2 X_{t-2}) + Var(\varepsilon_t) + 2\phi_1\phi_2 Cov(X_{t-1}, X_{t-2})$$
$$= Var(\phi_0) + \phi_1^2 Var(X_{t-1}) + \phi_2^2 Var(X_{t-2}) + \sigma^2 + 2\phi_1\phi_2 Cov(X_{t-1}, X_{t-2})$$
$$= \phi_1^2 Var(X_{t-1}) + \phi_2^2 Var(X_{t-2}) + \sigma^2 + 2\phi_1\phi_2 Cov(X_{t-1}, X_{t-2}) \qquad (4)$$

By the stationary conditions, we know

$$Var(X_t) = Var(X_{t-1}) = Var(X_{t-2}) \equiv \gamma_0$$

And,

$$Cov(X_t, X_{t-1}) = Cov(X_{t-1}, X_{t-2}) = Cov(X_{t-2}, X_{t-3}) \equiv \gamma_1$$

Euqation (4) could be rewritten as,

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + \sigma^2 + 2\phi_1\phi_2\gamma_1 \qquad (5)$$

So we need another equation associated with γ_0 and γ_1 to solve out the system.

And we need to find it from the ACF.

Go back to the original process,

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

Times both sides X_{t-k}

$$X_t X_{t-k} = \phi_0 X_{t-k} + \phi_1 X_{t-1} X_{t-k} + \phi_2 X_{t-2} X_{t-k} + \varepsilon_t X_{t-k}$$

Takes both hand side of the Expectation.

$$E(X_tX_{t-k}) = E(\phi_0 X_{t-k} + \phi_1 X_{t-1}X_{t-k} + \phi_2 X_{t-2}X_{t-k} + \varepsilon_t X_{t-k})$$

$$E(X_tX_{t-k}) = \phi_0 E(X_{t-k}) + \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) + E(\varepsilon_t X_{t-k})$$

Now we need to construct the Covariance of both hand side, use algebra.

$$E(X_tX_{t-k}) = \phi_0 E(X_{t-k}) + \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) + E(\varepsilon_t X_{t-k})$$

$$E(X_tX_{t-k}) - E(X_t)E(X_{t-k}) = \phi_0 E(X_{t-k}) + \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) + E(\varepsilon_t X_{t-k}) - E(X_t)E(X_{t-k})$$

$$Cov(X_t, X_{t-k}) = \phi_0 E(X_{t-k}) + \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) + E(\varepsilon_t X_{t-k}) - E(X_t)E(X_{t-k})$$

$$Cov(X_t, X_{t-k}) = \phi_0 \mu + \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) + 0 - E(X_t)E(X_{t-k})$$

$$\begin{aligned} Cov(X_t, X_{t-k}) = \phi_0 \mu + \phi_1 [E(X_{t-1}X_{t-k}) - E(X_{t-1})E(X_{t-k})] + \phi_2 [E(X_{t-2}X_{t-k}) - E(X_{t-2})E(X_{t-k})] \\ -\mu^2 + \phi_1 E(X_{t-1})E(X_{t-k}) + \phi_2 E(X_{t-2})E(X_{t-k}) \end{aligned}$$

Then plug in

$$\mu = E(X_{t-1}) = E(X_{t-2}) = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

The sum of the constant terms becomes 0. We got

$$Cov(X_t, X_{t-k}) = \phi_1 [E(X_{t-1}X_{t-k}) - E(X_{t-1})E(X_{t-k})] + \phi_2 [E(X_{t-2}X_{t-k}) - E(X_{t-2})E(X_{t-k})]$$

$$Cov(X_t, X_{t-k}) = \phi_1 Cov(X_t, X_{t-k}) + \phi_2 Cov(X_{t-2}, X_{t-k})$$

which yields,

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \quad \forall k \in \mathcal{N} \tag{6}$$

This is so-called auto-covariance function of the AR2 process.

If we consider a special case and plug in $k = 1$, the equation will hold as well.

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1}$$

Since

$$Cov(X_t, X_{t-1}) = \gamma_1 = \gamma_{-1} = Cov(X_{t-1}, X_t)$$

We have

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_{-1}$$
$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$$
$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0 \tag{7}$$

Combine with equation (5) and (7),

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + \sigma^2 + 2\phi_1\phi_2\gamma_1 \tag{5}$$
$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0 \tag{7}$$

We should be able to solve out the second (central) moments of AR2 process.

$$Var(X_t) = \gamma_0 = (1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1^2\phi_2}{1-\phi_2})^{-1}\sigma^2$$

And the Lag-1 Covariance

$$Cov(X_t, X_{t-1}) = \gamma_1 = (\frac{\phi_1}{1-\phi_2})(1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1^2\phi_2}{1-\phi_2})^{-1}\sigma^2$$

1.3 Covariance and Correlation Functions

By recursive formula equation (6)

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \quad \forall k \in \mathcal{N} \tag{6}$$

with γ_0 and γ_1 as initial conditions, we can find all γ_k.
For Autocorrelation Function, we know,

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

Divide both side of equation (6) by γ_0

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \quad \forall k \in \mathcal{N} \tag{6'}$$

Equation (6') is the so-called ACF of AR2 process, with ρ_0 and ρ_1 as initial conditions, we can find all ρ_k.